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GLOBAL DYNAMICS OF AN EPIDEMIC MODEL WITH CONSTANT IMMIGRANT AND NONLINEAR INCIDENT RATE

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Abstract

In this paper, we study the global dynamics of an epidemic model with constant immigrant and nonlinear incident rate. By carrying out the bifurcation analysis of the model, we show that there exist some values of the model parameters such that numerous kinds of bifurcation occur for the model, such as Hopf bifurcation, Bogdanov-Takens bifurcation.

1. Introduction

In the mathematical modeling of disease transmission, there is a classic model proposed by Kermack and McKendrick [6] in 1927.They divided the population being studied in time t into three classes labeled $S(t)$, $I(t)$ and $R(t)$, where $S(t)$ is the number of susceptible individuals,

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$I(t)$ is the number of infective individuals, and $R(t)$ is the number of removed individuals at time t, respectively. And they assumed that a rate of contacts by an infective with a susceptible is proportional to population size with constant of proportionality. It is clear that the assumption is too simple. Later Capasso and Serio [2] introduced a saturated incidence rate $g(I)S$ into the epidemic model after studying the cholera epidemic spread in Bari in 1973, which describes the contact between infective individuals and susceptible individuals, where

$$g(I) = \frac{kI}{1 + \alpha I}.$$

kI measures the infection force of the disease and $\frac{1}{(1 + \alpha I)}$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases. $g(I)$ tends to a saturation level when I gets large. The general incidence rate

$$\frac{kI^p S}{1 + \alpha I^q}$$

was proposed by Liu et al. [7, 8] and used by a number of authors, see, for example, [3, 4, 5, 1], etc. For a specific non-linear incident rate

$$\frac{kI^2 S}{1 + \alpha I^2}.$$

Ruan and Wang in [9] studied an epidemic model with this specific nonlinear incident rate and obtained lots of interesting dynamical behaviors of the model such as a limit cycle, two limit cycles and homoclinic loop, etc. Note that these functions $g(I)$ of the incident rate $g(I)S$ is monotone, which means that the contact rate between infective individuals and susceptible individuals is bigger and bigger as the number of infective individuals is getting larger. These functions $g(I)$ of the incident rate $g(I)S$ is monotone. Xiao and Ruan in [10] considered a special non- linear incident rate

$$\frac{kIS}{1+\alpha I^2},$$

where $g(I)$ is non-monotone. In [10], Xiao and Ruan presented the global analysis of an epidemic model with the non-monotonic incident rate and obtained either the number of infective individuals tends to zero as time evolves or the disease persists. Hence, the model cannot undergo any bifurcations. Zhou and Xiao in [11] considered a special nonlinear incident rate

$$\frac{kIS}{1+\beta I+\alpha I^2}.$$

They show that there exist some values of the model parameters such that numerous kinds of bifurcation occur for the model, such as Hopf bifurcation, Bogdanov-Takens bifurcation.

In this paper we consider the following SIRS model:

$$\begin{aligned}\frac{dS}{dt} &= aA - dS - \frac{kSI^2}{1+\alpha I^2} + \gamma R,\\ \frac{dI}{dt} &= \frac{kSI^2}{1+\alpha I^2} - (d+\mu)I,\\ \frac{dR}{dt} &= cA + \mu I - (d+\gamma)R,\end{aligned} \tag{1.1}$$

where $a + c = 1$ and a, c is a positive parameter. Where $S(t)$, $I(t)$ and $R(t)$ denote the numbers of susceptible, infective, and recovered individuals at time t, respectively, d is the natural death rate of the population, k is the proportionality constant, μ is the natural recovery rate of the infective individuals, y is the rate at which recovered individuals lose immunity and return to the susceptible class, α is a positive parameter.

2. Analysis for the Model

Before going into details, let us simplify this model. Summing up the three equations in (1.1) and denoting the number of total population by

$N(t)$, we obtain

$$\frac{dN}{dt} = A - dN.$$

Since $N(t)$ tends to a constant as t tends to infinity, we assume that the population is in equilibrium and investigate the behavior of the system on the plane $S + I + R = N_0 > 0$. Thus, we consider the reduced system

$$\frac{dI}{dt} = \frac{kI^2}{(1 + \alpha I^2)}(N_0 - I - R) - (d + \mu)I,$$

$$\frac{dR}{dt} = cA + \mu I - (d + \gamma)R. \tag{2.1}$$

To be concise in notations, re-scale (2.1) by

$$x = \sqrt{k / (d + \gamma)}I, \; y = \sqrt{k / (d + \gamma)}R,$$

and $\tau = (d + \gamma)t$. Then we obtain

$$\frac{dx}{d\tau} = \frac{x^2}{1 + px^2}(B - x - y) - mx,$$

$$\frac{dy}{d\tau} = n + qx - y, \tag{2.2}$$

where

$$p = \frac{\alpha(d + \gamma)}{k}, \; B = N_0\sqrt{\frac{k}{d + \gamma}}, \; m = \frac{d + \mu}{d + \gamma},$$

$$n = \sqrt{\frac{k}{(d + \gamma)^3}}cA, \; q = \frac{\mu}{d + \gamma}.$$

It can be seen that p, B, m, q, n are positive parameters.

In this paper, we focus on studying the existence of non-hyperbolic positive equilibria of (2.2) and their bifurcations. To find the positive equilibria of system (2.2), we set

$$\frac{x^2}{1+px^2}(B-x-y)-mx=0,$$

$$n+qx-y=0$$

which yields

$$(mp+q+1)x^2-(B-n)x+m=0.$$

Let $\Delta=(B-n)^2-4m(mp+q+1)$. Then we obtain the following lemma.

Lemma 1. (i) *System* (2.2) *has a unique positive equilibrium* $E^*(x^*, y^*)$ *if and only if the following conditions holds:*

$\Delta=0$, *and* $B-n>0$. *In this case,* $x^*=\dfrac{B-n}{2(mp+q+1)}$, $y^*=n+qx^*$.

(ii) *System* (2.2) *has two positive equilibria* $E_1(x_1, y_1)$ *and* $E_2(x_2, y_2)$ *if and only if* $\Delta>0$, $B-n>0$, *and* $1+q+pm>0$. *In this case,* $x_1=\dfrac{B-n-\sqrt{(B-n)^2-4m(mp+q+1)}}{2(mp+q+1)}$, $y_1=n+qx_1$; $x_2=\dfrac{B-n+\sqrt{(B-n)^2-4m(mp+q+1)}}{2(mp+q+1)}$, $y_2=n+qx_2$.

The Jacobian matrix of system (2.2) at equilibrium (x, y) is

$$M=\begin{pmatrix} \dfrac{x[B-n-(2+q)x-p(B-n)x^2+pqx^3]}{(1+px^2)^2} & \dfrac{-x^2}{1+px^2} \\ q & -1 \end{pmatrix}.$$

Therefore, the determinant of the matrix M is

$$\det M=\frac{x[n-B+2(1+q)x+p(B-n)x^2]}{(1+px^2)^2}.$$

Its sign is determined by

$$S_1 \triangleq n-B+2(1+q)x+p(B-n)x^2$$

and

$$\mathrm{tr}M = \frac{(qp - p^2)x^4 - p(B-n)x^3 - (2+q+2p)x^2 + (B-n)x - 1}{(1+px^2)^2}.$$

Its sign is determined by

$$S_2 \triangleq (qp - p^2)x^4 - p(B-n)x^3 - (2+q+2p)x^2 + (B-n)x - 1.$$

Note that $(mp+q+1)x^2 - (B-n)x + m = 0$, Then we have

$$r_1 = (mp+q+1)S_1$$

$$= [2mp + 4q + 2q^2 + 2mpq + 2 + p(B-n)^2]x - (B-n)(2mp+1+q),$$

$$r_2 = (mp+q+1)^3 S_2$$

$$= (B-n)[D_1(B-n)^2 + D_2]x + D_3(B-n)^2 + D_4,$$

where

$$D_1 = -p(mp+q+1),$$

$$D_2 = (mp+q+1)(2m^2p^2 + mp - 2p - 2qp - q - 1),$$

$$D_3 = mp(mp+p+1),$$

$$D_4 = (1+q)(mp+q+1)(2m^2p + mq + 2m - q - 1).$$

Theorem 2. (i) *The unique positive equilibrium* $E^*(x^*, y^*)$ *of system* (2.2) *is a degenerate equilibrium if* $\Delta = 0$ *and* $B - n > 0$, *where* $x^* = \dfrac{B-n}{2(mp+q+1)}$, $y^* = n + qx^*$.

(ii) *System* (2.2) *has two positive equilibria* $E_1(x_1, y_1)$ *and* $E_2(x_2, y_2)$ *if and only if* $\Delta > 0$, $B - n > 0$ *and* $1 + q + pm > 0$. *And further when* $r_2(x_2) = 0$, $E_1(x_1, y_1)$ *is a hyperbolic saddle,* $E_2(x_2, y_2)$ *is a center-type equilibrium, where*

$$x_1 = \frac{B - n - \sqrt{(B-n)^2 - 4m(mp+q+1)}}{2(mp+q+1)}, \; y_1 = n + qx_1,$$

$$x_2 = \frac{B - n + \sqrt{(B-n)^2 - 4m(mp+q+1)}}{2(mp+q+1)}, \; y_2 = n + qx_2,$$

This paper is organized as follows. In Section 3, there exists a set of values of parameters such that the model (2.2) has a there exists a set of values of parameters such that the model undergoes Bogdanov-Takens bifurcation of codimension 2 when two parameters vary in the small neighborhood of the set of parameter values. In Section 4, we show that there exist some values of parameters such that the model (2.2) has a positive equilibrium, which is a multiple focus of codimension 1. Choosing one parameter of the model as a bifurcation parameter, we discuss the Hopf bifurcation of the model. The paper ends with a brief conclusion.

3. Bogdanov-Takens Bifurcation

The purpose of this section is to study if there exist some values of model parameters such that model (2.2) undergoes the Bogdanov-Takens bifurcation. From Theorem 2, we know that the unique equilibrium $E^*(x^*, y^*)$ is degenerate if and only if (H1) $\Delta = 0$ and (H2) $B - n > 0$. In order to guarantee the existence of Bogdanov-Takens bifurcation, we further assume that (H3) $\operatorname{tr}(E_1) = 0$. Now we choose some values of parameters $n, p, q, m,$ and B such that (H1)-(H3) hold. Taking $m = 2, p = 1$ and $q = 5,$ we obtain that $B - n = 8, x^* = 1/2$ by (H1)-(H3). Taking $B = 9, n = 1,$ we obtain than $y^* = 7/2.$

Lemma 3. *When* $(m, p, q, B, n) = (2, 1, 5, 9, 1)$, *system* (2.2) *has a unique positive equilibrium* $(x^*, y^*) = (1/2, 7/2)$, *which is a cusp of codimension* 2.

Proof. Translating the unique positive equilibrium $(1/2, 7/2)$ into the origin, we set $X = x - 1/2, Y = y - 7/2,$ then (2.2) becomes

$$\frac{dX}{d\tau} = \frac{(X+1/2)^2(5-X-Y)}{1+(X+1/2)^2} - 2X - 1,$$

$$\frac{dY}{d\tau} = 5X - Y. \tag{3.1}$$

Using Taylor expansion to (3.1), we obtain

$$\frac{dX}{d\tau} = X - \frac{1}{5}Y - \frac{16}{25}XY + P(X, Y),$$

$$\frac{dY}{d\tau} = 5X - Y. \tag{3.2}$$

We set $x = X$, $y = X - Y/5$, then (3.2) becomes

$$\frac{dx}{d\tau} = y - \frac{16}{5}x^2 + \frac{16}{5}xy + \overline{P}(x, y),$$

$$\frac{dy}{d\tau} = -\frac{16}{5}x^2 + \frac{16}{5}xy + \overline{P}(x, y). \tag{3.3}$$

In order to obtain the canonical normal forms, we set $U = x - \frac{8}{5}x^2$, $V = y - \frac{16}{5}x^2$. then (3.3) becomes

$$\frac{dU}{d\tau} = V + R_1(U, V),$$

$$\frac{dV}{d\tau} = -\frac{16}{5}U^2 + \frac{16}{5}UV + R_2(U, V). \tag{3.4}$$

In the following we will find the universal unfolding of $E(1/2, 7/2)$ by choosing parameters m and q as bifurcation parameters in a small neighborhood of $(p, B, q, m, n) = (1, 9, 5, 2, 1)$.

Let $q = 5 + \lambda_1$, $m = 2 + \lambda_2$. And rewriting τ as t, then (2.2)

$$\frac{dx}{dt} = \frac{x^2}{1+x^2}(9 - x - y) - (2 + \lambda_1)x,$$

$$\frac{dy}{dt} = (5 + \lambda_2)x - y - 1. \tag{3.5}$$

If $\lambda_1 = 0$, $\lambda_2 = 0$, then $(1/2, 7/2)$ is a cusp of codimension 2 for (3.5).

Let $X = x - 1/2$, $Y = y - 7/2$. By the Taylor expansion, we have

$$\frac{dX}{dt} = -\frac{1}{2}\lambda_1 + (1 - \lambda_1)X - \frac{1}{5}Y - \frac{16}{25}XY + w_1(X, Y, \lambda),$$

$$\frac{dY}{dt} = \frac{1}{2}\lambda_2 + (5 + \lambda_2)X - Y, \tag{3.6}$$

where $\lambda = \lambda(\lambda_1, \lambda_2)$, $w_1(X, Y, \lambda)$ is a smooth function of X, Y and λ at least of order three in X and Y.

Set $x = X$, $y = -\frac{1}{2}\lambda_1 + (1 + \frac{3}{5}\lambda_1)X - \frac{1}{5}Y$. We get

$$\frac{dx}{dt} = y - \frac{16}{5}(1 + \frac{3}{5}\lambda_1)x^2 + \frac{16}{5}xy + w_2(x, y, \lambda),$$

$$\frac{dy}{dt} = a_0 + a_1 x + \frac{3}{5}\lambda_1 y + a_2 x^2 + a_3 xy + w_3(x, y, \lambda), \tag{3.7}$$

where $w_2(x, y, \lambda)$, $w_3(x, y, \lambda)$ is a smooth function of x, y and λ at least of order three in x and y, and

$$a_0 = -\lambda_1 - \frac{1}{10}\lambda_2,\ a_1 = -\frac{1}{5}\lambda_2 + \frac{3}{5}\lambda_1,$$

$$a_2 = -\frac{16}{5} - \frac{96}{25}\lambda_1 - \frac{144}{125}\lambda_1^2,\ a_3 = \frac{16}{5} + \frac{48}{25}\lambda_1.$$

Set $X = x - \frac{8}{5}x^2$, $Y = y - \frac{16}{25}(5 + 3\lambda_1)x^2 + w_2(x, y, \lambda)$. We get

$$\frac{dX}{dt} = Y,$$

$$\frac{dY}{dt} = a_0 + a_1 X + \frac{3}{5}\lambda_1 Y + b_1 X^2 + b_2 XY + w_4(X, Y, \lambda), \tag{3.8}$$

where $w_4(X, Y, \lambda)$ is a smooth function of X, Y and λ at least of order three in X and Y, and

$$b_1 = -\frac{16}{5} - \frac{8}{25}(\lambda_2 + 3\lambda_1),\ b_2 = -\frac{16}{25}(5 + 3\lambda_1).$$

Set $x = X + a_1 / 2b_1$, $y = Y$. We get

$$\frac{dx}{dt} = y,$$

$$\frac{dy}{dt} = e_0 + e_1 y + b_1 x^2 + b_2 xy + w_5(x, y, \lambda), \tag{3.9}$$

where $w_5(x, y, \lambda)$ is a smooth function of x, y and λ at least of order three in x and y, and

$$e_0 = \frac{4b_1 a_0 - a_1^2}{4b_1}, \; e_1 = \frac{6b_1\lambda_1 - 5a_1 b_2}{10b_1}.$$

Set $X = b_1 x$, $Y = -b_2 y$, $\tau = -\frac{b_1}{b_2} t$. We get

$$\frac{dX}{d\tau} = Y,$$

$$\frac{dY}{d\tau} = \tau_1 + \tau_2 Y + X^2 - XY + w_6(X, Y, \lambda), \tag{3.10}$$

where $w_6(X, Y, \lambda)$ is a smooth function of X, Y and λ at least of order three in X and Y, and

$$\tau_1 = \frac{b_2^2}{b_1} e_0, \; \tau_2 = -\frac{b_2}{b_1} e_1.$$

Let

$$J = \begin{pmatrix} \frac{\partial \tau_1}{\lambda_1} & \frac{\partial \tau_1}{\lambda_2} \\ \frac{\partial \tau_2}{\lambda_1} & \frac{\partial \tau_2}{\lambda_2} \end{pmatrix}.$$

And after simple calculation we can obtain that

$$\det(J)|_{\lambda=0} = -1 / 40 \neq 0.$$

Thus, by the Bogdanov and Takens bifurcation theorems, we obtain the following conclusion in a small neighborhood of $(\lambda_1, \lambda_2) = (0, 0)$.

Theorem 4. (i) *System* (3.5) *undergoes a saddle-node bifurcation, and the saddle-node bifurcation curve is* $SN = \{(\tau_1, \tau_2) : \tau_1 = 0, \tau_2 \neq 0\} = \{(\lambda_1, \lambda_2) : 1600\lambda_1 + 160\lambda_2 + 435\lambda_1^2 + 238\lambda_1\lambda_2 + 11\lambda_2^2 = 0\}$.

(ii) *System* (3.5) *undergoes a Hopf bifurcation, and there exist some values of* λ_1 *and* λ_2 *such that system* (3.5) *has a unique stable limit cycle.*

(iii) *System* (3.5) *undergoes a Homoclinic bifurcation, and there exist some other values of* λ_1 *and* λ_2 *such that system* (3.5) *has a unique stable homoclinic loop.*

4. Hopf Bifurcation

In this section, we will study the Hopf bifurcation of system (2.2) for some values of model parameters in the case: $\Delta > 0$, $B - n > 0$, and $\operatorname{tr}(E_2) = 0$, where $E_2(x_2, y_2)$ is a center-type equilibrium and

$$x_2 = \frac{B - n + \sqrt{(B-n)^2 - 4m(mp + q + 1)}}{2(mp + q + 1)}, \; y_2 = n + qx_2.$$

Now, we set $m = 2$, $p = 1$, $q = 9$, $B - n = 10$. Taking $B = 11$, $n = 1$, we obtain that $x_2 = 1/2$, $y_2 = 11/2$. We consider

$$\frac{dx}{d\tau} = \frac{x^2}{1 + x^2}(11 - x - y) - 2x,$$

$$\frac{dy}{d\tau} = 1 + 9x - y. \tag{4.1}$$

Lemma 5. $(1/2, 11/2)$ *is a stable weak focus of multiple one for system* (4.1)

Proof. We set $X = x - 1/2$, $Y = y - 11/2$, then (4.1) becomes

$$\frac{dX}{d\tau} = \frac{(X + 1/2)^2(5 - X - Y)}{1 + (X + 1/2)^2} 2X - 1,$$

$$\frac{dY}{d\tau} = 9X - Y. \tag{4.2}$$

Using Taylor expansion to (4.2), we obtain

$$\frac{dX}{d\tau} = X - \frac{1}{5}Y - \frac{16}{25}XY - \frac{16}{5}X^3 - \frac{16}{125}X^2Y + o(|X, Y|)^4,$$

$$\frac{dY}{d\tau} = 9X - Y. \tag{4.3}$$

Set $u = X$, $v = -\frac{5}{2\sqrt{5}} X + \frac{1}{2\sqrt{5}} Y$, $t = \frac{2\sqrt{5}}{5}\tau$. we get

$$\frac{du}{dt} = -v - \frac{8}{\sqrt{5}} u^2 - \frac{16}{5} uv - \frac{16}{25} u^2 v - \frac{96}{10\sqrt{5}} u^3 + o(|u, v|)^4,$$

$$\frac{dv}{dt} = u + 4u^2 + \frac{8}{\sqrt{5}} uv + \frac{8}{5\sqrt{5}} u^2 v + \frac{24}{5} u^3 + o(|u, v|)^4. \tag{4.4}$$

Thus, we can get the first Liapounov constant $w_1 = -106\sqrt{5}/25 < 0$. Therefore the origin is a stable weak focus of order one for (4.4). The conclusion follows.

In the following, we choose B as a bifurcation parameter.

Let $B = 11 + \varepsilon_1$. Then we may write (4.1) as follows

$$\frac{dx}{d\tau} = \frac{x^2}{1 + x^2}(11 + \varepsilon_1 - x - y) - 2x,$$

$$\frac{dy}{d\tau} = 1 + 9x - y. \tag{4.5}$$

The positive equilibrium of system (4.5) is $E^*\left(\frac{10 + \varepsilon_1 + \sqrt{\varepsilon_1^2 + 20\varepsilon_1 + 4}}{24}, \frac{90 + 9\varepsilon_1 + 9\sqrt{\varepsilon_1^2 + 20\varepsilon_1 + 4}}{24} + 1\right)$ and the linearizing matrix of (4.5) at E^* is

$$\mathbf{J} = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ 9 & -1 \end{pmatrix},$$

where

$$\Lambda_1 = \frac{-6}{(340 + \varepsilon_1^2 + 20\varepsilon_1 + 10\sqrt{\varepsilon_1^2 + 20\varepsilon_1 + 4} + \varepsilon_1\sqrt{\varepsilon_1^2 + 20\varepsilon_1 + 4})^2}$$

$$(-23600 + 40\varepsilon_1^3 + 552\varepsilon_1^2 + 3040\varepsilon_1 + \varepsilon_1^3\sqrt{\varepsilon_1^2 + 20\varepsilon_1 + 4}$$

$$+ 30\varepsilon_1^2\sqrt{\varepsilon_1^2 + 20\varepsilon_1 + 4} + 300\varepsilon_1\sqrt{\varepsilon_1^2 + 20\varepsilon_1 + 4} + \varepsilon_1^4 + 1000\sqrt{\varepsilon_1^2 + 20\varepsilon_1 + 4}).$$

$$\Lambda_2 = -\frac{52 + \varepsilon_1^2 + 20\varepsilon_1 + 10\sqrt{\varepsilon_1^2 + 20\varepsilon_1 + 4} + \varepsilon_1\sqrt{\varepsilon_1^2 + 20\varepsilon_1 + 4}}{340 + \varepsilon_1^2 + 20\varepsilon_1 + 10\sqrt{\varepsilon_1^2 + 20\varepsilon_1 + 4} + \varepsilon_1\sqrt{\varepsilon_1^2 + 20\varepsilon_1 + 4}}.$$

Therefore, the characteristic equation of which is

$$\lambda^2 + (1 - \Lambda_1)\lambda - \Lambda_1 - 9\Lambda_2 = 0.$$

Obviously, (i) $Re\lambda(\varepsilon_1)|(\varepsilon_1 = 0) = 0$, $Im\lambda(\varepsilon_1)|(\varepsilon_1 = 0) \neq 0$; (ii) $\frac{d\,\mathrm{Re}\,\lambda(\varepsilon_1)}{d\varepsilon_1}|(\varepsilon_1 = 0) \neq 0$; (iii) from Lemma 5, we have $w_1 < 0$.

Therefore, by Hopf bifurcation theory, we obtain the following result.

Lemma 6. *There exists a* $\sigma_1 > 0$, *and a function* $\varepsilon_1 = \varepsilon_1(x_1)$ *defined on* $0 < x_1 - \frac{1}{2} \leq \sigma_1$, *which satisfies* $\varepsilon_1(\frac{1}{2}) = 0$. *And when* $\varepsilon_1 = \varepsilon_1(x_1) < 0$, *system* (4.5) *has a unique stable limit cycle which passes through* $(x_1, 11/2)$.

5. Discussions

Epidemic mathematical models have become important tools to study the transmission dynamics of infectious diseases in host populations. In this paper, by combining qualitative and bifurcation analyses we have studied the global behavior of an epidemic model with constant immigrant and nonlinear incident rate. From the analysis, we have found that there exist some values of the model such that the model can undergo a series of bifurcations, such as saddle-node bifurcation, Hopf bifurcation and Bogdanov-Takens bifurcation. When a stable limit cycle surrounds the endemic equilibrium, it means that the number of the infective tends to a periodic function and the disease will exhibit frequently regular oscillation. Hence, the disease become periodic outbreak as time evolves. On the other hand, there exist some parameters values such that the model has two endemic equilibria (one is a saddle and the other is center-type equilibrium) and a stable homoclinic loop. Thus, the disease will persist.

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